BACK-PAPER ISI BANGALORE DECEMBER 2020

DIFFERENTIAL GEOMETRY 100 POINTS

Notes.
(a) This exam is for 3 hours.
(b) You may freely use any result proved in class. All other steps must be justified.
(c) $\mathbb{R}$ = real numbers.

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1. [20 points] Let $\gamma(t)$ be a unit-speed plane curve with nonzero curvature everywhere. Suppose there exists a constant $c \in \mathbb{R}$ such that $\gamma(t) \cdot N(t) = c$ where $N(t)$ denotes the principal normal to $\gamma(t)$. Prove the following
   (i) $\gamma(t)$ is a part of a circle centred at the origin.
   (ii) $c < 0$ and the radius of the circle in (i) is $-c$.

2. [20 points] Let C be the curve in $\mathbb{R}^2$ given by $2x^2 + 3y^2 = 1$. Calculate the value of the signed curvature κ_s at any point (x_0, y_0) on C. Find the maximum and minimum values of κ_s on C.

3. [20 points] Let S_1 and S_2 be the surfaces given by $\sin(x+y) - e^{y+z} + 1 = 0$ and $(x+1)^2 + y(z+1) - 1 = 0$ respectively. Verify that both S_1, S_2 are regular near the origin and that the intersection $S_1 \cap S_2$ is a regular curve C near the origin. Finally, find the tangent line to C at the origin.

4. [20 points] Let $U = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$, the upper half plane in $\mathbb{R}^2$. Suppose S is a regular surface in $\mathbb{R}^n$ and $\sigma \colon U \to S$ a chart such that the associated first fundamental form on U is
$$ds^2 = \frac{1}{v^2}(du^2 + dv^2).$$
Consider the line segments L_1, L_2, L_3 in U given by
$$L_1 = \{(0,b) \mid b \in [1,2]\}, \qquad L_2 = \{(a, a+1) \mid a \in [0,1]\}, \qquad L_3 = \{(a,2) \mid a \in [0,1]\}.$$
   (i) Calculate the length of the image curves $\sigma(L_i)$ on S.
   (ii) Calculate the angles between the curves $\sigma(L_i)$ and $\sigma(L_j)$ at their intersection points.
   (iii) Calculate the area of the region bounded by the three curves $\sigma(L_1), \sigma(L_2), \sigma(L_3)$.

5. [20 points] Let S be the torus, realised as a surface of revolution obtained by rotating about the z-axis, the profile curve in the x-z plane given by $(x-2)^2 + (z)^2 = 1$.
   (i) Find points p_1, p_2, p_3 on S such that the Gaussian curvature K satisfies $K(p_1) < 0$, $K(p_2) = 0$, $K(p_3) > 0$.
   (ii) At each of the points p_i, find the tangent plane, a unit normal and the principal curvatures of S at p_i.